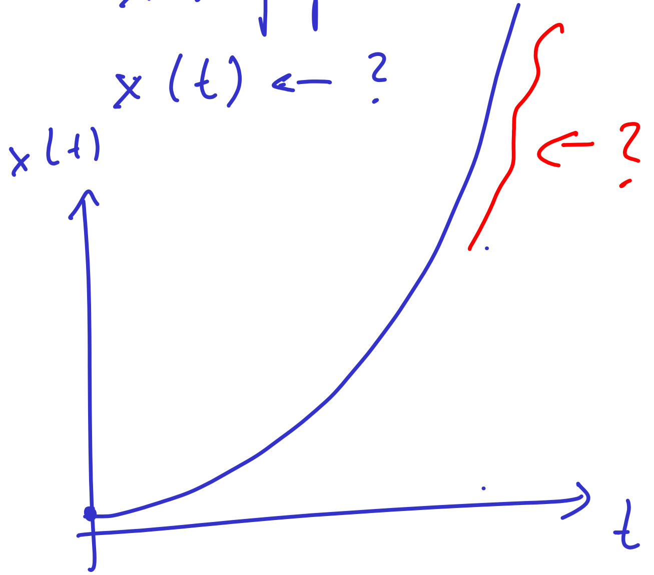
24-3-2020

Exponential growth : Growth rate is proportional to population

X : population

x (t) ← ?



$$\frac{dx}{dt} = \boxed{\dot{x} = r \cdot x}$$ Differential equation

Initial Value Problem (IVP)

Initial condition

$$x(t) = \underline{x_0} \cdot e^{rt} \longrightarrow \boxed{x(0) = x_0}$$

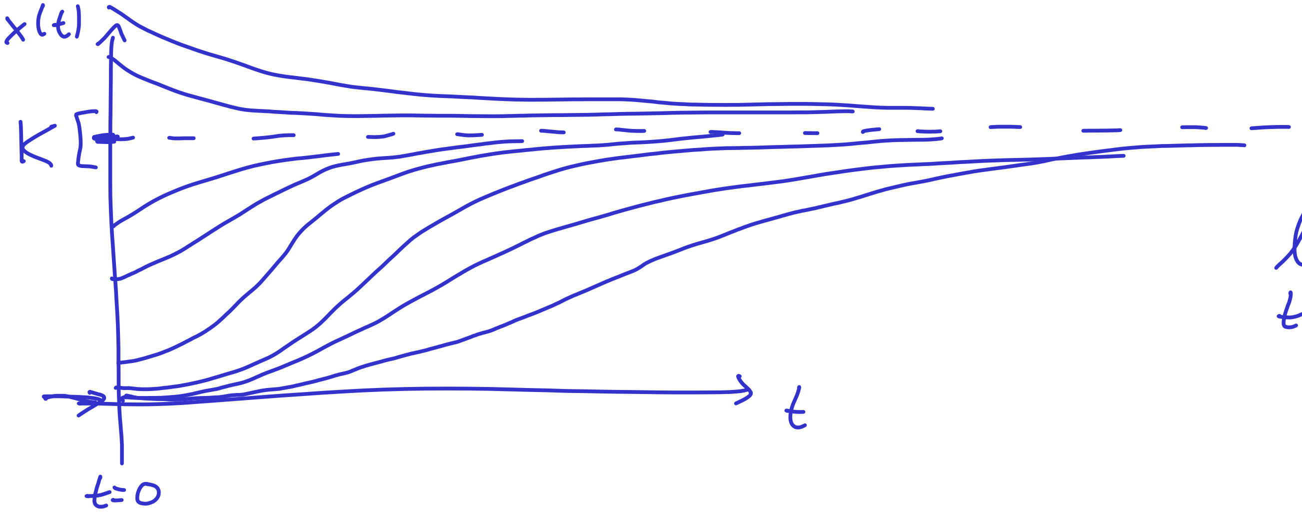
determined from initial conditions

Logistic growth (Verhulst 1838):

$$\longrightarrow \quad \dot{x} = r \cdot x \cdot \underbrace{\left(1 - \frac{x}{K}\right)}$$ K: capacity

$x \approx 0$: $1 - \frac{x}{K} \approx 1$ → approx. exponential growth

$x \approx K$: $1 - \frac{x}{K} \approx 0$ → no growth



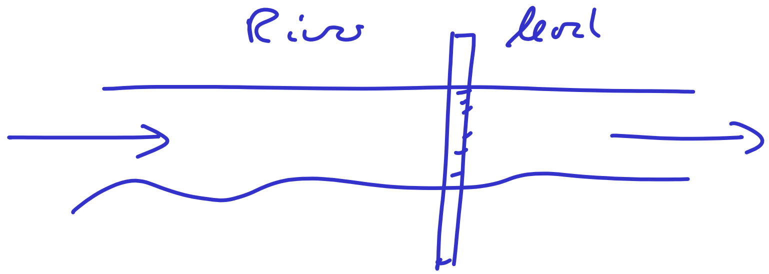
stable

$$\lim_{t \to \infty} x(t) = K$$

unstable

Assume that $\underline{x(0) = 0 \; : \; x(t) = 0}$

Steady-state (stationary state)

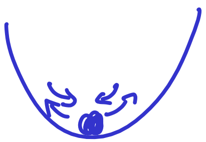
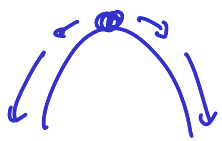


$$\xrightarrow{\text{birth}} X \xrightarrow{\text{death}}$$

Defined by $\dot{x} = 0$

$$\underbrace{\dot{x} = rx\left(1 - \frac{x}{K}\right) = 0}_{x = 0 \text{ or } x = K}$$

What are the steady states?

stable  unstable 

General differential equation (1-dim):

$$\boxed{\dot{x} = f(x)}$$ autonomous system

$\Big(f(x) = r \cdot x \leftarrow$ exp. growth

$f(x) = rx\left(1 - \frac{x}{k}\right) \leftarrow$ log. growth

$\vdots$

$\Big)$

Steady-states: $\dot{x} = 0$

$\Leftrightarrow f(x) = 0$

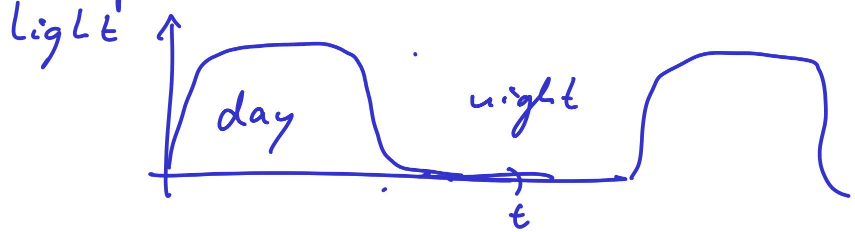
non-autonomous system

$$\boxed{\dot{x} = f(x, t)}$$

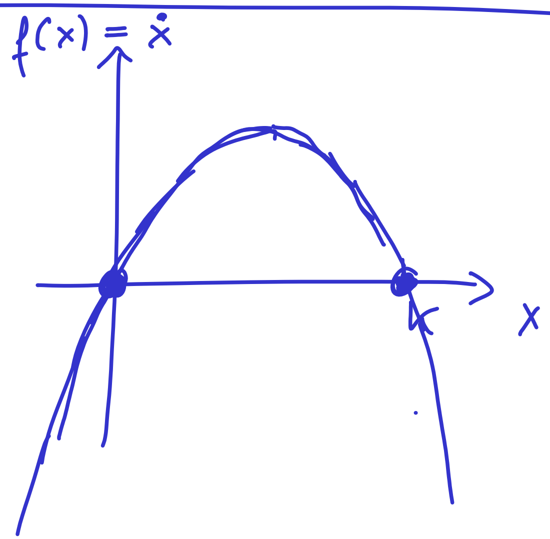
time

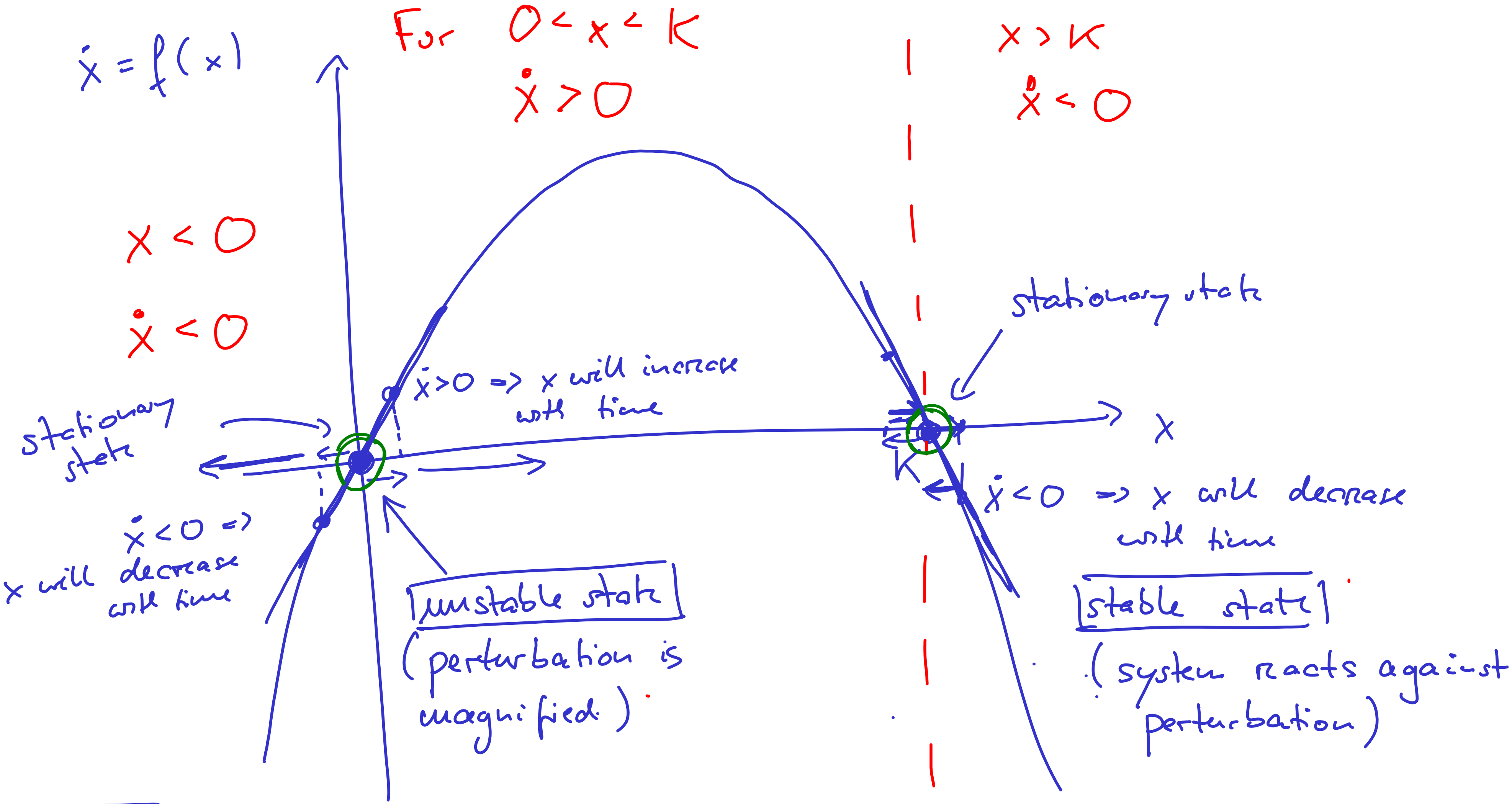
e.g. circadian clock

dependence on time



$f(x) = rx\left(1 - \frac{x}{k}\right)$





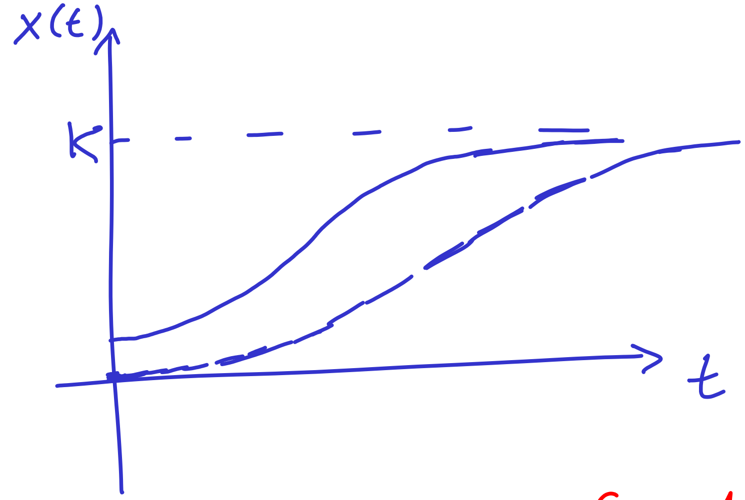
$$\boxed{\dot{x} = f(x)}$$

Stability of a steady state is determined by the sign of the derivative of $f(x)$

$f'(x) > 0$: unstable

$f'(x) < 0$: stable

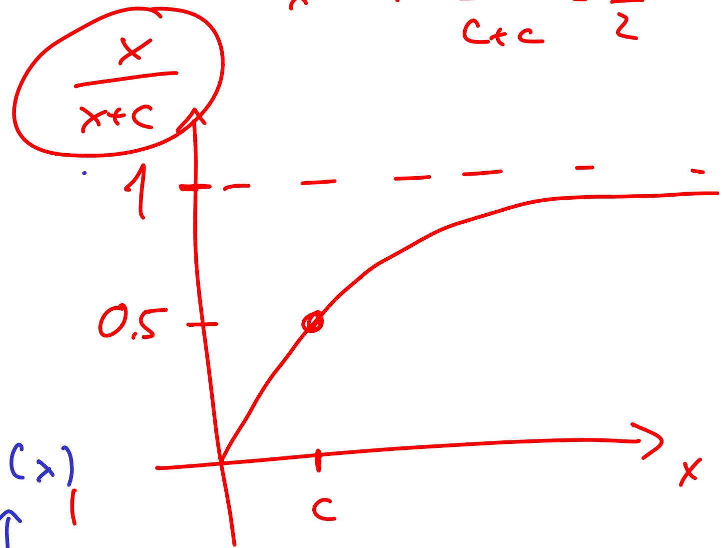
$\dot{x} = r\cdot x\cdot\left(1-\frac{x}{K}\right)$ $\qquad K > x > 0 : \dot{x} > 0$



Application to macroscopic ecosystems
(sexually reproducing species):

- the behaviour for small population sizes is unrealistic

$x = c$: $\frac{c}{c+c} = \frac{1}{2}$

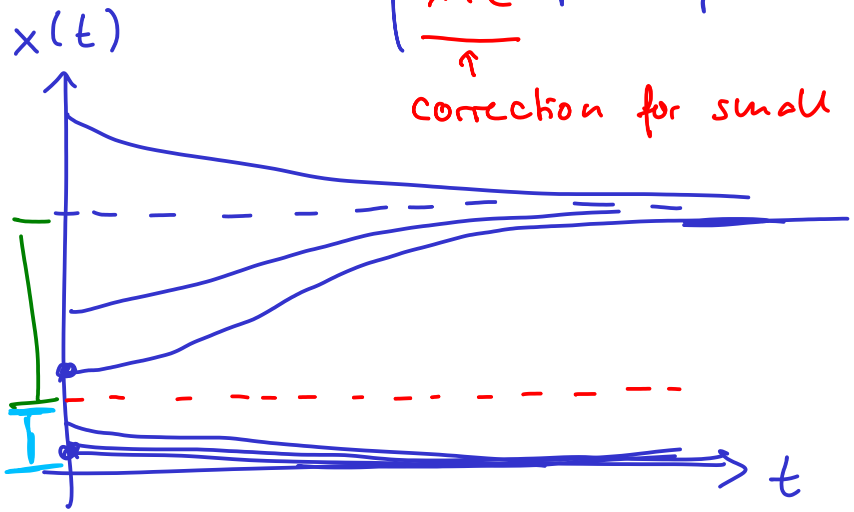


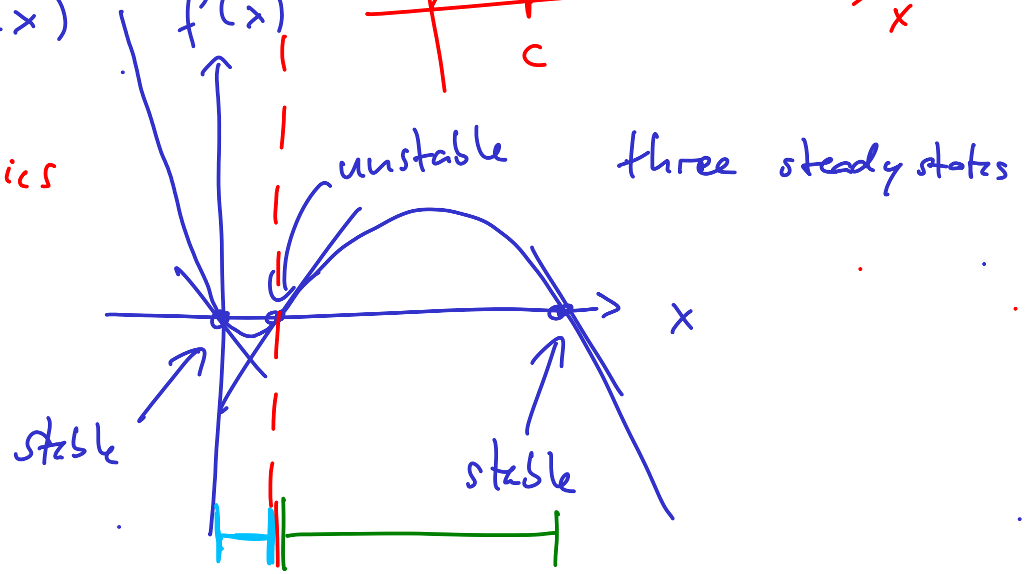
Net growth rate (constant): $r = b - d$

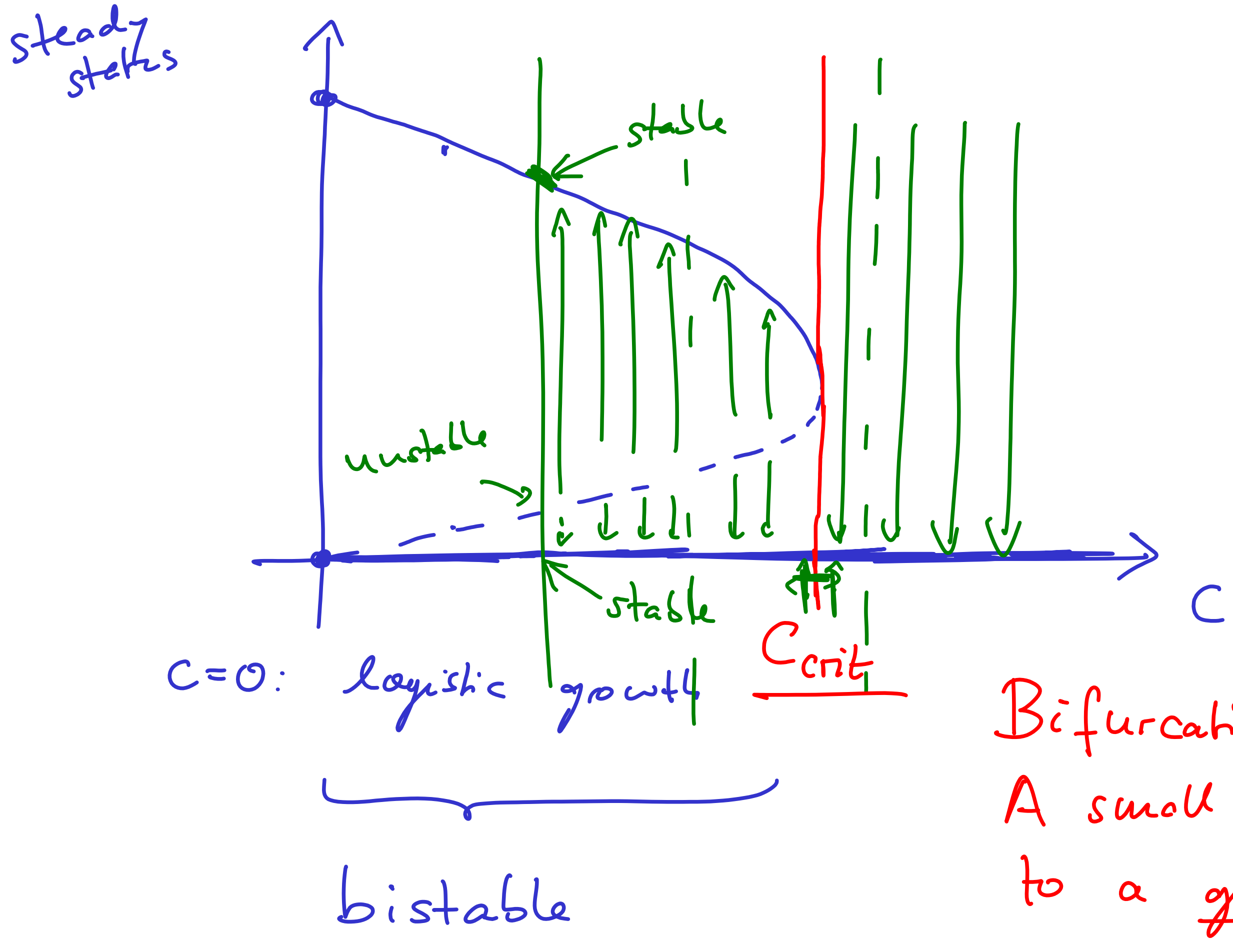
net growth, birth, death

$$\dot{x} = r\cdot x\left(\frac{x}{x+c}\overbrace{\frac{b}{r} - \frac{d}{r}}^{1} - \frac{x}{K}\right) = f(x)$$

correction for small population densities







$C < C_{crit}$:

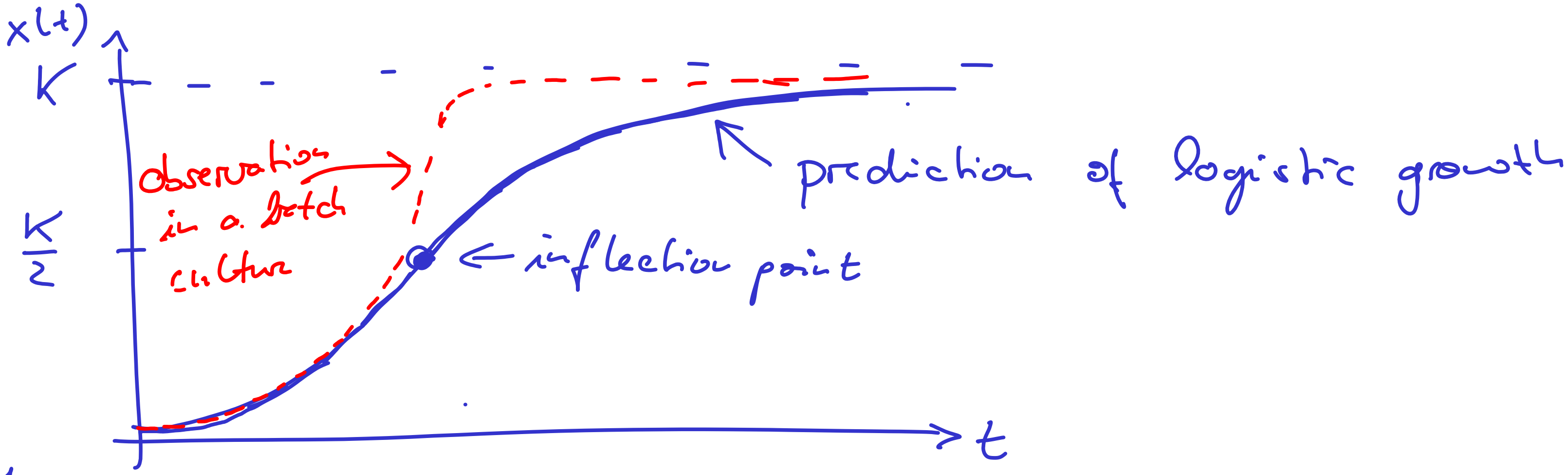
2 stable steady-states
1 unstable

$C > C_{crit}$:

1 stable steady-state
($x = 0$, extinction)

Bifurcation:

A small change of a parameter leads to a <u>global</u> change of the system's behaviour



Jacques Monod (1950): bacterial growth law

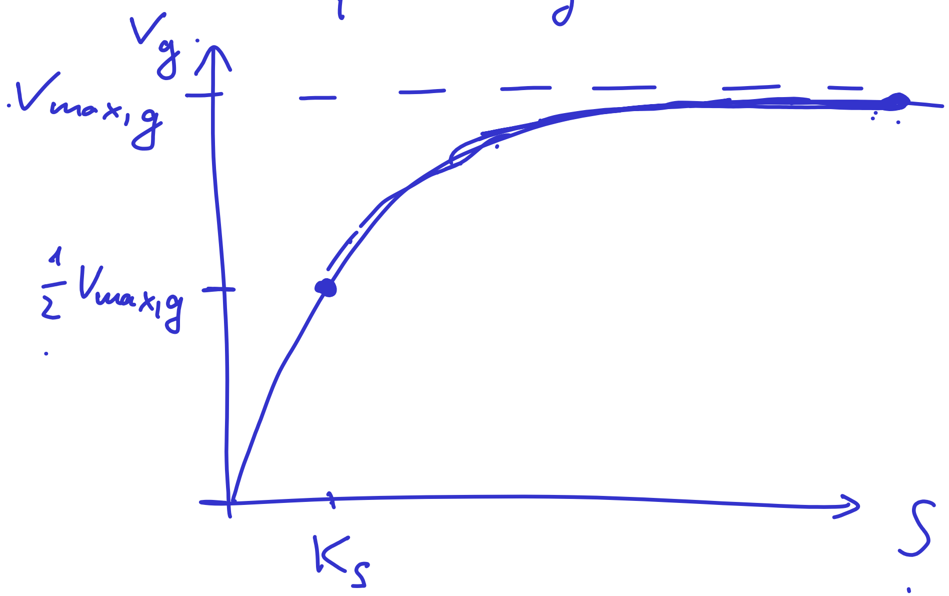
Growth rate of microbes depends on the concentration of a "limiting nutrient"

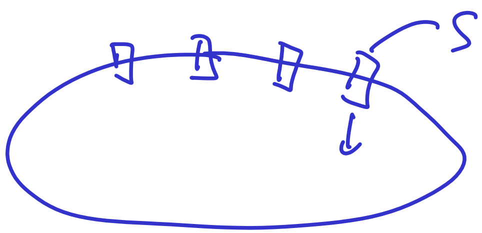
$$V_g = V_{max,g} \cdot \frac{S}{K_S + S}$$

concentration of limiting substrate

growth rate

Monod-constant





$$\dot{x} = \frac{dX}{dt} = V_g = r_g \cdot X$$

$$r_g = r_{max,g} \cdot \frac{S}{K_s + S}$$

X : population of bacteria

S : substrate

↳ $\dot{S} = ?$

$$\boxed{\dot{X} = r_g \cdot X = r_{max,g} \cdot X \cdot \frac{S}{K_s + S}}$$

growth rate of bacteria depends on

a) population x , b) substrate concentration

$$\boxed{\dot{S} = - \frac{1}{Y_{X/S}} r_g \cdot X}$$

yield factor: how much biomass do you get from the nutrients?

a common unit for biomass and nutrient is

C-mol

common yield factors for E. coli / yeast under aerobic growth exceed 0.5 (0.5 - 0.7)

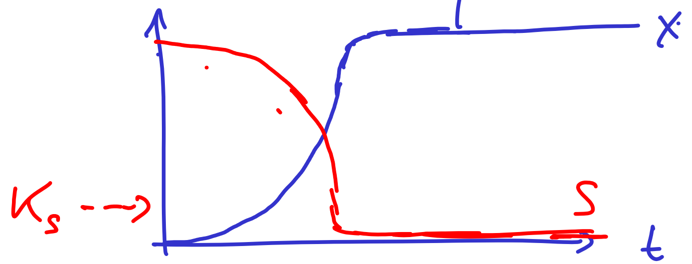
Two coupled differential equations!

$$\boxed{\begin{aligned} \dot{X} &= r_{max,g} \cdot X \cdot \frac{S}{K_s + S} \\ \dot{S} &= - \frac{1}{Y_{X/S}} \cdot r_{max,g} \cdot X \cdot \frac{S}{K_s + S} \end{aligned}}$$

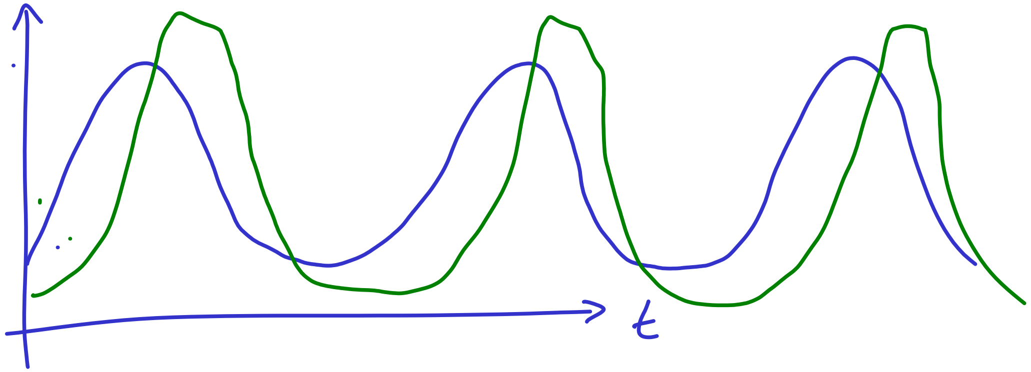
Species → $\dot{X}$

resource → $\dot{S}$

} 2-dimensional system



Lotka-Volterra System
(1920) (1926).



Predator-Prey model

x : prey (e.g. rabbit)

y : predator (e.g. fox)

$$\begin{aligned} \dot{x} &= r_1 x - a_1 \cdot x \cdot y \\ \dot{y} &= a_2 \cdot x \cdot y - r_2 \cdot y \end{aligned}$$

2-dim

Generalisation

Generalised Lotka-Volterra Systems

x_i, $i = 1..n$: n different species

$$\boxed{\dot{x}_i = r_i \cdot x_i + \sum_{j=1}^{n} a_{ij} \underbrace{x_i x_j}_{\text{interaction term}}}$$

n-dim

r_i can be positive or negative

a_{ij} : interaction terms / interaction matrix

can be positive or negative

Consumer - Resource Models (MacArthur, 1970)

X_i , $i = 1..n$: species

R_j , $j = 1..m$: resources

"maintenance"

$$\dot{X}_i = \underbrace{b_i}_{\text{yield}} X_i \left(\sum_{j=1}^{m} \underbrace{c_{ij}}_{\text{"capture rate"}} \underbrace{w_j}_{\text{"value of resource } j\text{"}} R_j - m_i \right)$$

$$\dot{R}_j = R_j \left(\underbrace{r_j \left(1 - \frac{R_j}{K_j} \right)}_{\text{influx}} - \underbrace{\sum_{i=1}^{n} c_{ij} X_i}_{\text{consumption}} \right)$$