PS-Model

The following equations are from population biology (Edelstein-Keshet, 1988):

$$\frac{dP}{dt} = -S = f(P,S) \quad (1)$$
$$\frac{dS}{dt} = \alpha P(1-P) - S = g(P,S) \quad (2)$$

with $-\infty < P < +\infty, -\infty < S < +\infty$ and $\alpha > 0$. Analyze the system's dynamic behavior:
Determine the stability points. For which values of α are they stable?

stationary points $E^* = (P^*, S^*)$
P^* =equilibrium concentration of P
S^* =equilibrium concentration of S

$$\frac{dP}{dt} = \frac{dS}{dt} = 0$$

$$\frac{dP}{dt} = -S = 0 \Rightarrow S^* = 0$$
$$\frac{dS}{dt} = \alpha P(1-P) - S = 0$$
$$\Rightarrow \alpha P(1-P) = 0$$
$$\Rightarrow P(1-P) = 0$$
$$\Rightarrow P_1^* = 0, P_2^* = 1$$
$$\Rightarrow E_1^* = (0,0), E_2^* = (1,0)$$

stability

Calculation of the Jacobian:

$$J = \begin{pmatrix} \frac{\delta f(P,S)}{\delta P} & \frac{\delta f(P,S)}{\delta S} \\ \frac{\delta g(P,S)}{\delta P} & \frac{\delta g(P,S)}{\delta S} \end{pmatrix}$$

Eigenvalues:
$det(J - \lambda E) = 0$
$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} -\lambda & -1 \\ \alpha(1-2P) & -1-\lambda \end{vmatrix}$$

$$\begin{aligned} &= -\lambda \cdot (-1-\lambda) - (-1) \cdot \alpha \cdot (1-2P) \\ &= \lambda^2 + \lambda + \alpha \cdot (1-2P) = 0 \end{aligned}$$

$$\Rightarrow \lambda_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha \cdot (1-2P)}$$

Thus, for E_1^* follows:

$$\begin{aligned} \lambda_{11} &= -\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha} \\ \lambda_{12} &= -\frac{1}{2} - \sqrt{\frac{1}{4} - \alpha} \end{aligned}$$

For E_2^* follows:

$$\begin{aligned} \lambda_{21} &= -\frac{1}{2} + \sqrt{\frac{1}{4} + \alpha} \\ \lambda_{22} &= -\frac{1}{2} - \sqrt{\frac{1}{4} + \alpha} \end{aligned}$$

In $\mathbb{R}$ a stationary point is stable if the eigenvalues are smaller than 0.
Thus, E_1^* is stable for $\alpha \leq \frac{1}{4}$.
E_2^* is not stable for $\lambda \in \mathbb{R}$.